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The Convergence of the Differential Transform Method for Solving Semi-Explicit Index-2 Differential Algebraic Systems

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Abstract

This paper investigates the convergence of the differential transform method for solving differential algebraic systems. Differential algebraic systems, which combine differential and algebraic equations, pose significant challenges due to their inherent structural constraints and index-related complexities. The study presents a comprehensive convergence analysis of differential transform method, emphasizing the conditions required for the method to produce accurate and reliable solutions. Additionally, the multi-stage differential transform method is employed to extend the convergence interval, thereby enhancing the method's applicability to more complex systems. Numerical examples illustrate the effectiveness of the proposed approach, demonstrating its accuracy and computational efficiency in solving differential algebraic systems. The results confirm that the differential transform method, along with its multi-stage variant, is a powerful tool for obtaining approximate solutions to differential algebraic systems while preserving their inherent properties.

Keywords: differential transform method; differential algebraic systems; multi-stage differential transform method; convergence analysis; index complexity; approximate solutions; numerical methods; computational efficiency.

1 Introduction

Differential algebraic systems (DASs) are an important class of mathematical models that consist of both differential and algebraic equations. These systems frequently arise in various scientific and engineering applications, including electrical circuits, control systems, and mechanical systems with constraints [16, 23]. Additional studies have also highlighted their significance in modeling complex physical phenomena [21, 25]. Solving DASs is challenging due to the interplay between the differential and algebraic components, which often leads to index-related complications and stringent consistency conditions. Consequently, the development of efficient numerical methods that ensure both convergence and accuracy remains a critical area of research [17, 8].

Several approaches have been proposed to address these challenges, including advanced series expansion techniques [4, 1] and stability-oriented computational algorithms [9]. Related studies on direct multistep methods have also provided effective strategies for handling delay differential equations with boundary and initial value problems, offering insights that are valuable for the broader class of constrained dynamical systems [12]. The differential transform method (DTM) has gained recognition as a powerful tool for solving differential equations, including fractional and nonlinear types, because it can produce series solutions without requiring discretization [15, 2]. It has also been successfully applied to various engineering and scientific problems [14].

Extending DTM to differential algebraic systems offers a promising way to manage the complexities inherent in such problems, as it transforms the original system into a simpler algebraic form, enabling more straightforward handling of both the differential and algebraic components. However, ensuring the convergence of DTM for DASs particularly for higher-index systems requires careful theoretical consideration [26, 24]. Prior investigations have examined specific criteria for convergence and stability [5]. This study focuses on analyzing the convergence of DTM when applied to DASs, identifying the theoretical framework and conditions under which the method is reliable. The analysis also incorporates the role of the multi-stage differential transform method (MsDTM) in extending the convergence interval, which is particularly advantageous for systems with complex dynamic behavior.

Through illustrative examples, the paper demonstrates the practical effectiveness of DTM and MsDTM in solving DASs while maintaining accuracy and computational efficiency. The remainder of the paper is structured as follows: Section 2 provides an overview of DASs and the fundamental principles of DTM. Section 4 introduces the proposed approach for solving nonlinear DASs. Section 7 presents the convergence analysis and the conditions required for reliability. Section 8 contains numerical examples to validate the theoretical findings. Finally, Section 9 concludes with key insights and suggestions for future research directions.

2 Differential Transform Method (DTM)

The Differential Transform Method (DTM) is a semi-analytical technique used to obtain approximate solutions of differential equations in the form of a power series. It is based on the Taylor series expansion but transforms the problem into a set of recursive algebraic relations. DTM is particularly effective for solving initial value problems involving linear and nonlinear differential or differential-algebraic equations. This section introduces the core definitions and properties of DTM that form the basis for its application.

Definition 2.1. [27, 3] *If a function $v(t)$ is analytic in domain of interest, its differential transform can be* *expressed as:*

$$V(n) = \frac{1}{n!} \left. \frac{d^n v(t)}{dt^n} \right|_{t=t_0}. \tag{1}$$

Here, $V(n)$ denotes the transformed form of $v(t)$ at $t = t_0$.

Definition 2.2. *According to [27, 3], the inverse differential transform reconstructs $v(t)$ from the transformed set $\{V(n)\}_{n=0}^{\infty}$ as:*

$$v(t) = \sum_{n=0}^{\infty} V(n)(t - t_0)^n. \tag{2}$$

This summation provides the original function as a power series expansion around t_0.

If (1) is substituted into (2), the resulting expression is

$$v(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{d^n v(t)}{dt^n} \right|_{t=t_0} (t - t_0)^n. \tag{3}$$

The fundamental concept of the DTM method is based on expanding functions as power series, as outlined in Definitions 2.1 and 2.2. To illustrate the application of DTM in solving a system of ordinary differential equations (ODEs), consider the nonlinear system given by

$$\frac{dv\left(t\right)}{dt}=f\left(v\left(t\right),t\right), \qquad t\geq t_0, \tag{4}$$

subject to the initial condition,

$$v(t_0)=v_0. \tag{5}$$

Using DTM, an approximate solution to (4) is formulated as

$$v\left(t\right)=\sum_{n=0}^{\infty} V\left(n\right)\left(t-t_0\right)^n. \tag{6}$$

By applying the method to both the given initial condition (4) and the governing (5), the unknown coefficients $V\left(0\right), V\left(1\right), V\left(2\right), \ldots$ can be determined. This leads to the following results:

$$V\left(0\right)=v_0, \tag{7}$$

and

$$(1+n)\,V\left(n+1\right)=F\left(V\left(0\right),\ldots,V\left(n\right),n\right), \qquad n=0,1,2,\ldots. \tag{8}$$

Here, the differential of $f\left(v\left(t\right),t\right)$ is represented as $F(V\left(0\right),\ldots,V\left(n\right),n)$. From (7) and (8), the sequence $V\left(n\right)$, for $n=0,1,2,\ldots$ is calculated, yielding the final solution,

$$v\left(t\right)=\sum_{n=0}^{m} V\left(n\right)\left(t-t_0\right)^n. \tag{9}$$

Thus, (4)–(5) yield an exact solution as a direct result of (6).

The terms $V(n)$ and $U(n)$ correspond to the transform functions $v(t)$ and $u(t)$. The fundamental operations of the Differential Transform Method (DTM) are summarized in Table 1.

Table 1: Primary computational rules of the DTM.

Initial Function	**Transformed Function**
$\alpha v(t) \pm \beta u(t)$	$\alpha V(k) \pm \beta U(k)$
$v(t)u(t)$	$\sum_{r=0}^{n} V(r)\,U(n-r)$
$v(t)u(t)r(t)$	$\sum_{r=0}^{n}\sum_{l=0}^{r} V(l)\,U(r-l)\,R(n-r)$
$\frac{d^n}{dt^n}[v(t)]$	$(k+1)(k+2)\dots(k+n)\,V(k+n)$
$e^{\mu t}$	$\frac{\mu^n e^{\mu t_0}}{n!}$
$\sin(\phi t)$	$\frac{\phi^n}{n!}\sin\left(\phi t_0 + \frac{\pi n}{2}\right)$
$\cos(\phi t)$	$\frac{\phi^n}{n!}\cos\left(\phi t_0 + \frac{\pi n}{2}\right)$

A recursive formulation for the unknowns $V(0), V(1), V(2), \dots$ is obtained by applying the Differential Transform Method (DTM) to the initial conditions given in (4) and (5). Although the DTM converges only over small intervals, this motivates the use of a technique that enlarges the convergence domain. This new technique is the Multi-stage Differential Transform Method (MsDTM), which will be illustrated in the next section.

2.1 Characteristics of the Differential Transform Method (DTM)

The Differential Transform Method (DTM) possesses several attractive characteristics, which makes it an effective tool for solving ordinary and differential-algebraic equations.

- **Semi-analytical approach:** DTM provides a solution in the form of a power series, preserving the analytical structure while reducing computational complexity.
- **No discretization, linearization, or perturbation:** The method avoids traditional numerical techniques such as discretization or linearization, preserving the original nonlinear behavior of the system.
- **High accuracy with fewer terms:** DTM often achieves accurate approximations using only a few terms of the series.
- **Efficient handling of initial conditions:** Initial conditions are naturally incorporated in the recurrence relations without extra computation.

- **Suitable for nonlinear problems:** DTM handles nonlinear terms efficiently through its recursive structure.
- **Reduced computational burden:** Compared to classical numerical methods, DTM reduces the complexity of solving high-order or coupled systems by converting them into algebraic forms.
- **Extendable to various problem types:** DTM has been successfully applied to ordinary differential equations (ODEs), partial differential equations (PDEs), integro-differential equations, and differential-algebraic equations (DAEs), including fractional-order systems.

3 Problem Formulation

We consider a nonlinear differential-algebraic system of **index-2**, which consists of a set of coupled differential and algebraic equations of the form,

$$\begin{cases} Mv'(t) & = f(t, v(t), \lambda(t)), \\ g(t, v(t)) & = 0, \end{cases} \qquad t \in [0, T],$$

subject to the initial condition:

$$v(0) = v_0,$$

where

- $v(t) \in \mathbb{R}^n$ is the vector of differential variables,
- $\lambda(t) \in \mathbb{R}^r$ is the vector of Lagrange multipliers associated with the algebraic constraints,
- $M \in \mathbb{R}^{n\times n}$ is a constant singular or nonsingular mass matrix,
- $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$ is a sufficiently smooth nonlinear function,
- $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^r$ represents the algebraic constraints.

This system is said to be of **index-2** because differentiating the algebraic constraint $g(t, v(t)) = 0$ twice is necessary to obtain an explicit expression for $\lambda(t)$ in terms of $v(t)$ and its derivatives.

The objective of this study is to:

- Apply the Differential Transform Method (DTM) to derive approximate analytical solutions for index-2 DAEs.
- Perform a rigorous convergence analysis to establish the validity and reliability of the method.
- Employ the Multi-stage Differential Transform Method (MsDTM) to extend the convergence interval and enhance computational efficiency.
- Demonstrate the effectiveness of the approach through numerical experiments involving nonlinear index-2 DAEs.

This formulation reflects the inherent difficulties associated with index-2 systems and motivates the development of robust semi-analytical methods for their efficient and accurate solution.

4 Solving Nonlinear Systems of Differential Algebraic Equations

Assume the formula of the system of nonlinear index-2 DAEs is as follows:

$$\begin{cases} M(v).v' &= f(v,v') - G^T(v).\lambda, \\ \quad 0 &= g(v), \end{cases} \tag{10}$$

where v is the vector of differential state variables, $M(v)$ is a square coefficient matrix (depending on v), $f(v,v')$ is a vector of known functions of v and v', $G(v) = \dfrac{dg(v)}{dv}$ is the Jacobian matrix of the constraint functions $g(v)$, and $v(t_0) = v_0$ are the initial conditions.

Applying DTM to both sides of system (10),

$$DT[M(v).v']_{n-1} = DT[f(v,v') - G^T(v).\lambda]_{n-1}, \quad n \geq 1, \tag{11}$$

then, the following is obtained based on the properties of the power series and Adomain polynomials mentioned in [6, 7]:

$$\begin{cases} \sum_{l=0}^{n-1} M_{n-1-l}.(l+1)V_{l+1} &= f_{n-1} - \sum_{l=0}^{n-1} G^T_{n-1-l}.\lambda_l, \\ \qquad 0 &= g_n, \end{cases} \tag{12}$$

for $l = n-1$, then the following equation is obtained:

$$M_0 n V_n = f_{n-1} - \sum_{l=0}^{n-2} M_{n-1-l}.(l+1)V_{l+1} - G_0^T\lambda_{n-1} - \sum_{l=0}^{n-2} G^T_{n-1-l}.\lambda_l. \tag{13}$$

From (13) the following equation is obtained:

$$M_0 n V_n + G_0^T\lambda_{n-1} = f_{n-1} - \sum_{l=0}^{n-2} \left[M_{n-1-l}.(l+1)V_{l+1} + G^T_{n-1-l}.\lambda_l\right], \tag{14}$$

$$nV_n + M_0^{-1}G_0^T\lambda_{n-1} = M_0^{-1}\left[f_{n-1} - \sum_{l=0}^{n-2}\left[l(l+1)M_{n-1-l}V_{l+1} + G^T_{n-1-l}.\lambda_l\right]\right]. \tag{15}$$

Assume $G_0V_n = S_n$, then $S_n = -g_n + G_0V_n, n \geq 1$. Hence, the following equation is obtained:

$$G_0M_0^{-1}G_0^T\lambda_{n-1} = -nS_n + G_0M_0^{-1}\left[f_{n-1} - \sum_{l=0}^{n-2}[l(l+1)M_{n-1-l}V_{l+1} + G^T_{n-1-l}.\lambda_l]\right]. \tag{16}$$

From (16), λ_{n-1} is calculated as follows:

$$\lambda_{n-1} = (G_0M_0^{-1}G_0)^{-1}(-nS_n + G_0r_{n-1}), \quad n \geq 1, \tag{17}$$

where $r_{n-1} = f_{n-1} - \sum_{l=0}^{n-2}[l(l+1)M_{n-1-l}V_{l+1} + G^T_{n-1-l}.\lambda_l]$.

Then, from (13), V_n is evaluated,

$$V_n = \frac{1}{n}\left[-M_0^{-1}G_0^T\lambda_{n-1} + r_{n-1}\right]. \tag{18}$$

System (10) is converted into an algebraic system after applying the DTM method. Finally, the inverse transform function is used to obtain the approximate solution as follows:

$$\begin{cases} v(t) &= \sum_{k=0}^{K} v_k t^k, \\ \lambda(t) &= \sum_{k=0}^{K} \lambda_k t^k. \end{cases} \tag{19}$$

5 Algorithm of Solving Nonlinear System of Differential Algebraic Equations

An algorithm for solving nonlinear system of index-2 DAEs is proposed as follows:

Input:

- K: Order of approximation for the power series solution.
- n, $r < n$: Dimensions of the matrices.
- $M \in \mathbb{R}^{n\times n}$: Coefficient matrix.
- $f \in \mathbb{R}^n$, $g \in \mathbb{R}$: Given functions.
- $v_0 = v(0) \in \mathbb{R}^n$: Initial value.

Output:

- Approximate solutions: $v(t) = \sum_{k=0}^{K} V_k t^k, \quad \lambda(t) = \sum_{k=0}^{K} \lambda_k t^k.$

6 Algorithm Steps

Initialization:

- Compute the domain polynomials M_k, G_k, f_k, and g_k for $k = 0, 1, \dots, K$.

Compute S_k:

- For $k = 1, 2, \dots, K$, compute:

$$S_k = -g_k + G_0 V_k, \quad \text{where} \quad G_0 = g'(v_0).$$

Iterative computation:

- For $k = 1$ to K, do:
  (i) Compute the intermediate vector r_{k-1}:

$$r_{k-1} = M_0^{-1} \left(f_{k-1} - \sum_{l=0}^{k-2} \left(G_{k-1-l}^T \lambda_l + M_{k-1-l} g_l \right) \right).$$

(ii) Solve for λ_{k-1} from the linear system:

$$\left(G_0 M_0^{-1} G_0^T\right) \lambda_{k-1} = G_0 r_{k-1} + k \sum_{l=0}^{k-1} G_{k-l} V_l.$$

(iii) Compute:

$$g_{k-1} = -M_0^{-1} G_0^T \lambda_{k-1} + r_{k-1}.$$

(iv) Compute:

$$V_k = \frac{1}{k} g_{k-1}.$$

Construct approximate solutions:

$$v(t) = \sum_{k=0}^{K} V_k t^k, \quad \lambda(t) = \sum_{k=0}^{K} \lambda_k t^k.$$

7 Convergence Analysis of Differential Transform Method

This section presents the convergence analysis of the DTM approaches when applied to systems of DAEs. Theorems and lemmas related to convergence are introduced. Previous studies have examined the convergence of semi-analytic methods, including the DTM and its fractional extension (FDTM) [19, 18]. Similar analyses have also been conducted for the Adomian Decomposition Method (ADM) [20, 10] and the Homotopy Perturbation Method (HPM) [22, 13]. Building on these results, a convergence study of the multi-stage DTM (Ms-DTM) for DAEs, particularly in the context of non-autonomous systems, is also presented. A convergence study of the multi-stage DTM approach for DAEs as non-autonomous systems is offered based on the aforementioned results. The following are the key features of MsDTM as a tool for resolving non-autonomous systems of DAEs. The approximative solution is first derived as a power series,

$$v(t) = \sum_{i=0}^{\infty} V(i)(t - t_0)^i, \tag{20}$$

where $V(t)$ is the differential transform of $v(t)$, after applying DTM over each sub-interval. The key idea behind multi-stage DTM is to get an approximative solution in the form of a power series,

$$v(t) = \sum_{i=0}^{\infty} V(i)(t - t_0)^i, \quad t \in I,$$

where

$$I = (t_0, T).$$

Theorem 7.1. [11] *Consider the functions $v_{n(x,t)}$ and $v(x,t)$ defined within the Banach space $(C[I], \|.\|)$. The series solution $\sum_{n=1}^{\infty} v(n) t^n$, as formulated in (20), converges under the condition $0 \leq \Omega \leq 1$.*

Proof. Let S_n represent the sequence of partial sums of the series (20). To establish that S_n is a Cauchy sequence within the Banach space $(C[I], \|.\|)$, consider the following:

$$\|S_{n+1}(t) - S_{n(t)}\| = \|v_{n+1}(t)\| \leq \Omega\|v_{n(t)}\| \leq \Omega^2\|v_{n-1}(t)\| \leq \ldots \leq \Omega^{n+1}(t)\|v_0(t)\|. \tag{21}$$

For arbitrary integers $n, m \in \mathbb{N}$,$n \geq m$, applying triangular inequality iteratively gives

$$\begin{aligned} \|S_n - S_m\| &= \|(S_{n(t)} - S_{n-1}(t)) + (S_{n-1}(t) - S_{n-2}(t)) + \ldots + (S_{m+1}(t) - S_{m(t)})\| \\ &\leq \|S_n(t) - S_{n-1}(t)\| + \|S_{n-1}(t) - S_{n-2}(t)\| + \ldots + \|S_{m+1}(t) - S_{m(t)}\| \\ &\leq \Omega^n\|v_0(t)\| + \Omega^{n-1}\|v_0(t)\| + \ldots + \Omega^{m+1}\|v_0(0)\|. \end{aligned} \tag{22}$$

Since the sum of this geometric sequence satisfies,

$$\leq (\Omega^{m+1} + \Omega^{m+2} + \ldots)\|v_0(0)\| = \frac{\Omega^{m+1}}{1-\Omega}\|v_0(0)\|, \quad n \geq m, \quad |\Omega| < 1,$$

it follows that,

$$\lim_{n,m\to\infty} \|S_n - S_m\| = 0. \tag{23}$$

Thus, $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence in $(C[I], \|.\|)$, implying the series $\sum_{n=0}^{\infty} v_n(t)$, converges, complete the proof. □

Definition 7.1. *For any $i \in \mathbb{N} \cup \{0\}$, define:*

$$\Omega_j = \begin{cases} \dfrac{\|v_{j+1}\|}{\|v_j\|}, & \text{if } \|v_j\| \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

Corollary 7.1. *In Theorem 7.1, the condition $\sum_{j=0}^{\infty} v(j)$ guarantees convergence to an exact solution v, provided $0 \leq \Omega_j \leq 1$ for all $j = 1, 2, 3, \ldots$*

Corollary 7.2. *For cases where v_j and $\tilde{v}$ correspond to standard and multistage DTM methods, respectively, if Ω_j's is consistently smaller than $\tilde{\Omega}_j$'s, the rate of convergence of $\sum_{j=0}^{\infty} v(j)$ to the exact solution surpasses that of $\sum_{j=0}^{\infty} \tilde{v}(j)$. Furthermore, if $\tilde{\Omega}_i < \Omega_i$ for all j, it implies that the convergence rate of $\sum_{j=0}^{\infty} \tilde{v}(j)$ to the exact solution is at least as fast as that of $\sum_{j=0}^{\infty} v(j)$.*

8 Numerical Examples

The two examples are solved using the technique proposed in this section.

Example 8.1. *The following system of nonlinear index-2 DAEs is given by:*

$$\begin{cases} u_1' &= u_2^2 - 2u_1\lambda, \\ u_2' &= u_1u_2 + 2u_2\lambda, \\ 0 &= u_1^2 - u_2^2 + 1. \end{cases} \tag{25}$$

The system is equipped with the initial values $u_1(0) = 0$, $u_2(0) = 1$, and the exact solutions are

$$u_1(t) = \tan(t), \quad u_2(t) = \sec(t), \quad \lambda(t) = 0.$$

Furthermore,

$$M(u) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad f(u) = \begin{pmatrix} u_2^2 \\ u_1u_2 \end{pmatrix}, \quad g(u) = u_1^2 - u_2^2 + 1,$$

and the Jacobian of the function $g(u)$ in (25) is

$$G(u) = \begin{pmatrix} 2u_1 & -2u_2 \end{pmatrix}.$$

The full-row rank is $r = 1$. The MsDTM is applied to the interval $[0, T] = [0, 6]$ with an approximation order $K = 6$, and the interval is subdivided into $N = 200$ parts. The computational results are shown in Figures 1–6. Figures 1, 3, and 5 show the solution of components u_1, u_2, and λ, while Figures 2, 4, and 6 illustrate the corresponding errors.

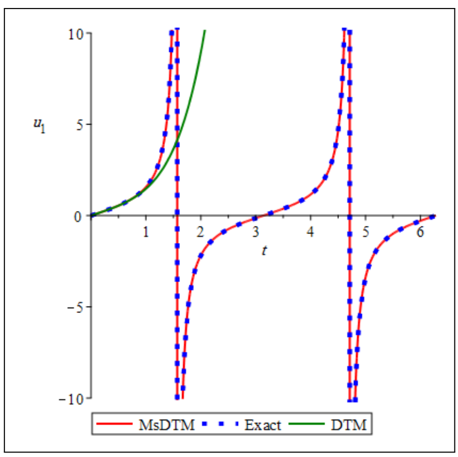

Figure 1: Approximate solution of MsDTM, DTM and exact solution of u_1 of (25).

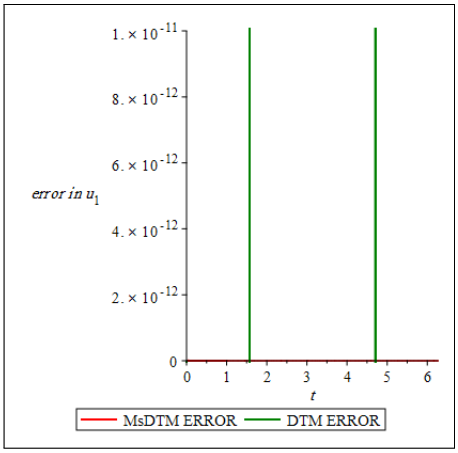

Figure 2: MsDTM Error and DTM error of u_1 of (25).

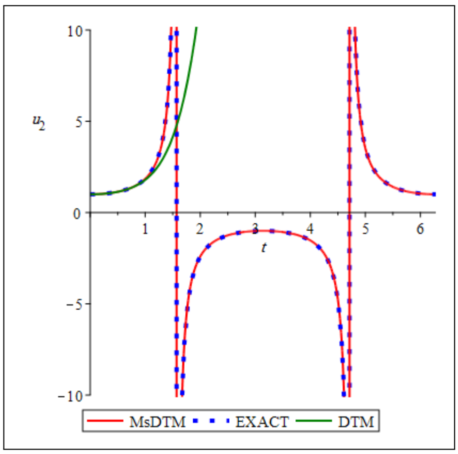

Figure 3: Approximate solution of MsDTM, DTM and exact solution of u_2 of (25).

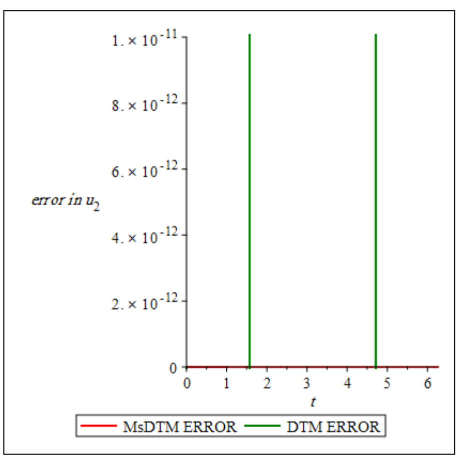

Figure 4: MsDTM Error and DTM error of u_2 of (25).

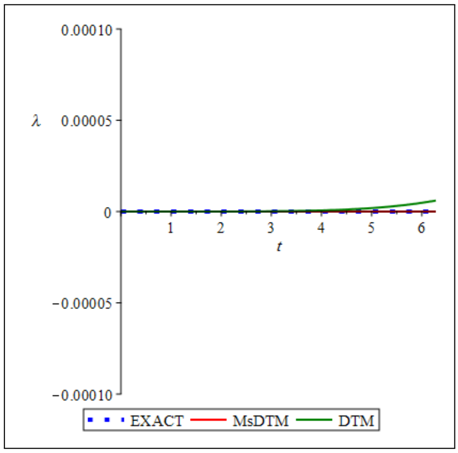

Figure 5: Approximate solution of MsDTM, DTM and exact solution of λ of (25).

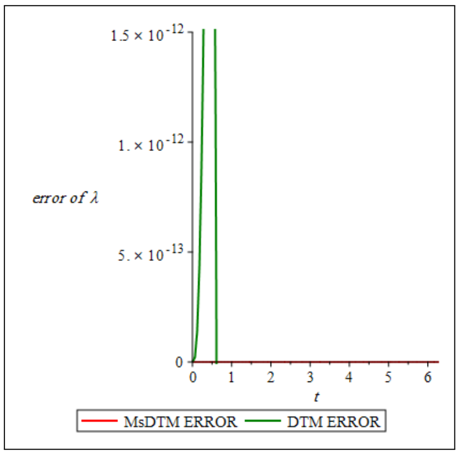

Figure 6: MsDTM Error and DTM error of λ of (25).

The absolute error of the components u_1 and u_2 for system (25), obtained using MsDTM with step size $h = \dfrac{T}{N} = 0.03$, order $K = 6$ and $N = 200$. These represent the errors relative to the exact solutions of the components u_1, u_2 and λ consecutively. The numerical results show that approximate solutions of components u_1, u_2, and λ agree excellently with exact solutions of the same components.

8.1 Convergence analysis of Example 8.1

Consider a system of nonlinear index-2 DAEs as follows:

Example 8.2.

$$\begin{cases} v_1' &= v_1 + v_1^3 v_2 - 2v_1 v_2 \lambda, \\ v_2' &= v_1^4 v_2 - 2v_2 - v_1^2 \lambda, \\ 0 &= v_1^2 v_2 - 1, \end{cases} \tag{26}$$

with initial values $v_1(0) = 1$, $v_2(0) = 1$, and exact solutions as follows:

$$v_1(t) = e^t, \quad v_2(t) = e^{-2t}, \quad \lambda(t) = e^{2t}.$$

The approximate solution after applying DTM over interval $I = [0, 6]$ is obtained as follows:

$$v(t) = \sum_{k=0}^{\infty} V(k) t^k = \begin{bmatrix} 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \dots \\ 1 - 2t + 2t^2 - \frac{4}{3}t^3 + \frac{2}{3}t^4 + \dots \end{bmatrix}, \tag{27}$$

$$\lambda(t) = \sum_{k=0}^{\infty} \lambda(k) t^k = 1 + 2t + 2t^2 + \frac{4}{3}t^3 + \frac{2}{3}t^4 + \dots. \tag{28}$$

The Ω_i's are computed to approximate the series solution, which remains in agreement with the exact solution over the range $I = [0, 6]$. The seies expansion of the estimated expression for the variable v_1 is given as follows:

$$\begin{cases} v_{10} &= 1, \\ v_{11} &= 1 + t, \\ v_{12} &= 1 + t + \frac{1}{2}t^2, \\ v_{13} &= 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3, \\ v_{14} &= 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4, \\ & \vdots \\ v_{1K}. \end{cases} \tag{29}$$

$$\begin{cases} y_{11} &= v_{11} - v_{10} = t, \\ y_{12} &= v_{12} - v_{11} = \frac{1}{2!}t^2, \\ y_{13} &= v_{13} - v_{12} = \frac{1}{3!}t^3, \\ y_{14} &= v_{14} - v_{13} = \frac{1}{4!}t^4, \\ & \vdots \\ y_{1K}. \end{cases} \tag{30}$$

$$\begin{cases} \Omega_1 &= \dfrac{\|y_{12}\|}{\|y_{11}\|} = \dfrac{\sqrt{3}}{6} \leq 1, \\ \Omega_2 &= \dfrac{\|y_{13}\|}{\|y_{12}\|} = \dfrac{\sqrt{3}}{9} \leq 1, \\ \Omega_3 &= \dfrac{\|y_{14}\|}{\|y_{13}\|} = \dfrac{\sqrt{3}}{12} \leq 1, \\ &\vdots \\ \Omega_K &= \dfrac{\|y_{1,k+1}\|}{\|y_{1k}\|} \leq 1. \end{cases} \tag{31}$$

When the power series solution is an infinite series, Theorem 7.1 defines the norm as a maximum norm. All Ω_i's for components v_2 and λ over interval $I = [0, 6]$ are computed as follows, based on a similar method:

$$\begin{cases} \Omega_1 &= \dfrac{\|y_{22}\|}{\|y_{21}\|} = \dfrac{\sqrt{3}}{3} \leq 1, \\ \Omega_2 &= \dfrac{\|y_{23}\|}{\|y_{22}\|} = \dfrac{2\sqrt{3}}{9} \leq 1, \\ \Omega_3 &= \dfrac{\|y_{24}\|}{\|y_{23}\|} = \dfrac{\sqrt{3}}{6} \leq 1, \\ &\vdots \\ \Omega_K &= \dfrac{\|y_{2,k+1}\|}{\|y_{2k}\|} \leq 1. \end{cases} \tag{32}$$

For component λ, the following is obtained:

$$\begin{cases} \Omega_1 &= \dfrac{\|y_{32}\|}{\|y_{31}\|} = \dfrac{\sqrt{3}}{3} \leq 1, \\ \Omega_2 &= \dfrac{\|y_{33}\|}{\|y_{32}\|} = \dfrac{2\sqrt{3}}{9} \leq 1, \\ \Omega_3 &= \dfrac{\|y_{34}\|}{\|y_{33}\|} = \dfrac{\sqrt{3}}{6} \leq 1, \\ &\vdots \\ \Omega_K &= \dfrac{\|y_{3,k+1}\|}{\|y_{3k}\|} \leq 1. \end{cases} \tag{33}$$

According to Corollary 7.1, any value of Ω_i which is less than or equal to 1, over the time interval $I = [0, 6]$, the power series of the approximation solution for all components converge.

Example 8.3. *Consider a system of nonlinear index-2 DAEs as follows:*

$$\begin{cases} u_1' &= u_2 - 2u_1u_2^2 - 2u_1\lambda, \\ u_2' &= -2u_1^3 - u1 - 2u_2\lambda, \\ 0 &= u_1^2 + u_2^2 - 1, \end{cases} \tag{34}$$

with initial values $u_1(0) = 0$, $u_2(0) = 1$, and exact solutions as follows:

$$u_1(t) = \sin(t), \quad u_2(t) = \cos(t), \quad \lambda(t) = -\cos^2(t).$$

Whereas

$$M(u) = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad f(u) = \begin{pmatrix} u_2 - 2u_1u_2^2 \\ -2u_1^3 - u_1 \end{pmatrix}, \quad g(u) = u_1^2 + u_2^2 - 1, \quad G(u) = \begin{pmatrix} 2u_1 & 2u_2 \end{pmatrix},$$

is full row rank $r = 1$. The proposed technique is applied on a system (34) over the interval $[0, T] = [0, 6]$, the order approximation $K = 6$, and the number of subdivisions $N = 300$. Figures 7, 9, and 11 show the solutions of components u_1, u_2 and λ. The errors of the components u_1, u_2 and λ are presented in Figures 8, 10 and 12. It is clear from the displaced results that there is good agreement between the approximate solutions and the exact solutions of the components u_1, u_2, and λ.

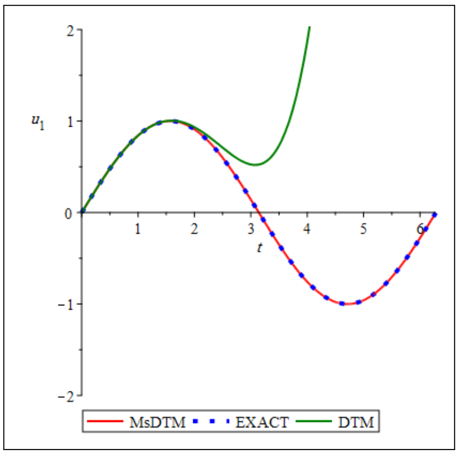

Figure 7: Approximate solution of MsDTM, DTM and exact solution of u_1 of (34).

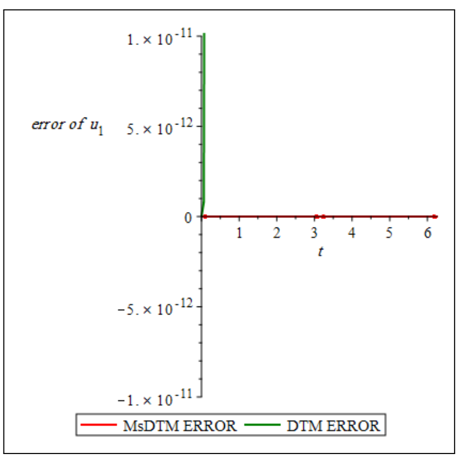

Figure 8: MsDTM error and DTM error of u_1 of (34).

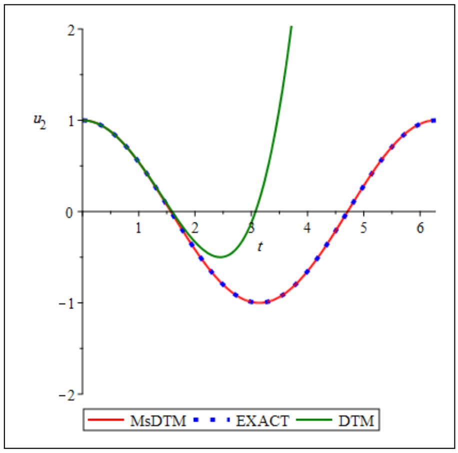

Figure 9: Approximate solution of MsDTM, DTM and exact solution of u_2 of (34).

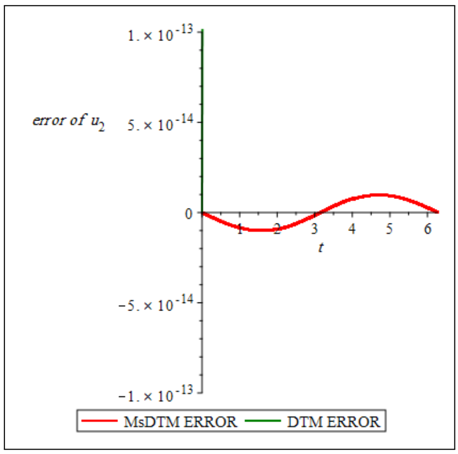

Figure 10: MsDTM error and DTM error of u_2 of (34).

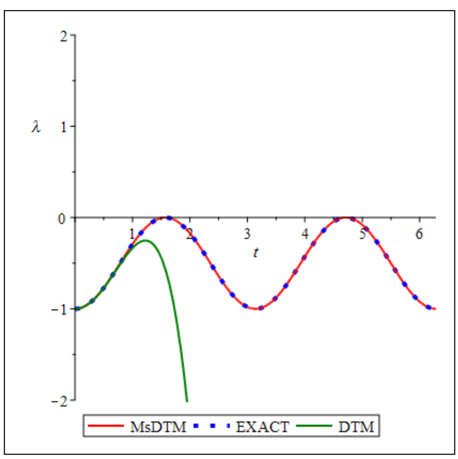

Figure 11: Approximate solution of MsDTM, DTM and exact solution of λ of (34).

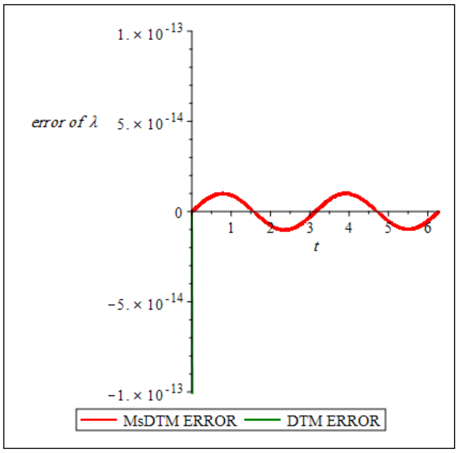

Figure 12: MsDTM error and DTM error of λ of (34).

Absolute error of components u_1 and u_2 obtained for system (34) using MsDTM with step size $h = \dfrac{T}{N} = 0.02$, $K = 6$, $N = 300$ and Eu_1, Eu_2, $E\lambda$ are exact solutions of the components u_1, u_2 and λ consecutively.

8.2 Convergence analysis of Example 8.3

Consider a system of nonlinear index-2 DAEs as follows:

Example 8.4.

$$\begin{cases} v_1' &= v_2 - 2v_1v_2^2 - 2v_1\lambda, \\ v_2' &= -2v_1^3 - v_1 - 2v_2\lambda, \\ 0 &= v_1^2 + v_2^2 - 1, \end{cases} \tag{35}$$

with initial values $v_1(0) = 0, v_2(0) = 1$, *and exact solutions as follows:*

$$v_1(t) = \sin(t), \quad v_2(t) = \cos(t), \quad \lambda(t) = -\cos^2(t).$$

The estimate solution after applying DTM over the range $I = [0, 6]$ is obtained as follows:

$$v(t) = \sum_{k=0}^{\infty} V(k)t^k = \begin{bmatrix} t - \frac{1}{6}t^3 + \frac{1}{120}t^5 + \frac{1}{5040}t^7 + \dots \\ 1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 + \frac{1}{720}t^6 + \frac{1}{40320}t^8 + \dots \end{bmatrix}, \tag{36}$$

$$\lambda(t) = \sum_{k=0}^{\infty} \lambda(k)t^k = -1 + t^2 - \frac{1}{3}t^4 + \frac{2}{45}t^6 + \dots. \tag{37}$$

The Ω_i's are computed to approximate the series solution, which is convergent to the exact solution over $I = [0, 6]$. The power series of the approximate solution for the variable v_1 is obtained as follows:

$$\begin{cases} v_{10} &= 0, \\ v_{11} &= t, \\ v_{12} &= t + \frac{1}{6}t^3, \\ v_{13} &= t + \frac{1}{6}t^3 - \frac{1}{120}t^5, \\ v_{14} &= t + \frac{1}{6}t^3 - \frac{1}{120}t^5 + \frac{1}{5040}t^7, \\ v_{15} &= t + \frac{1}{6}t^3 - \frac{1}{120}t^5 + \frac{1}{5040}t^7 - \frac{1}{362880}t^9, \\ & \vdots \\ v_{1K}. \end{cases} \tag{38}$$

$$\begin{cases} y_{11} &= v_{11} - v_{10} = t, \\ y_{12} &= v_{12} - v_{11} = -\frac{1}{6}t^3, \\ y_{13} &= v_{13} - v_{12} = \frac{1}{120}t^5, \\ y_{14} &= v_{14} - v_{13} = -\frac{1}{5040}t^7, \\ & \vdots \\ y_{1K}. \end{cases} \tag{39}$$

$$\begin{cases} \Omega_1 &= \frac{\|y_{12}\|}{\|y_{11}\|} = \frac{\sqrt{5}}{30} \leq 1, \\ \Omega_2 &= \frac{\|y_{13}\|}{\|y_{12}\|} = \frac{\sqrt{5}}{100} \leq 1, \\ \Omega_3 &= \frac{\|y_{14}\|}{\|y_{13}\|} = \frac{\sqrt{5}}{210} \leq 1, \\ & \vdots \\ \Omega_K &= \frac{\|y_{1,K+1}\|}{\|y_{1K}\|}. \end{cases} \tag{40}$$

When the power series solution is an infinite series, Theorem 7.1 defines the norm as a maximum norm. All Ω_i's for components v_2 and λ over interval $I = [0, 6]$ are computed as follows, based on a similar method:

$$\begin{cases} \Omega_1 &= \dfrac{\|y_{22}\|}{\|y_{21}\|} = \dfrac{\sqrt{5}}{60} \leq 1, \\ \Omega_2 &= \dfrac{\|y_{23}\|}{\|y_{22}\|} = \dfrac{\sqrt{5}}{150} \leq 1, \\ \Omega_3 &= \dfrac{\|y_{24}\|}{\|y_{23}\|} = \dfrac{\sqrt{5}}{280} \leq 1, \\ & \vdots \\ \Omega_K &= \dfrac{\|y_{2,K+1}\|}{\|y_{2K}\|}. \end{cases} \tag{41}$$

For component λ, the following is obtained:

$$\begin{cases} \Omega_1 &= \dfrac{\|y_{32}\|}{\|y_{31}\|} = \dfrac{\sqrt{5}}{15} \leq 1, \\ \Omega_2 &= \dfrac{\|y_{33}\|}{\|y_{32}\|} = \dfrac{2\sqrt{5}}{75} \leq 1, \\ \Omega_3 &= \dfrac{\|y_{34}\|}{\|y_{33}\|} = \dfrac{\sqrt{5}}{70} \leq 1, \\ & \vdots \\ \Omega_K &= \dfrac{\|y_{3,K+1}\|}{\|y_{3K}\|}. \end{cases} \tag{42}$$

According to Corollary 7.1, any Ω value which is either less than or equal to 1, the power series of the approximation solution for each component converges over the time range $[0, 6]$.

9 Conclusion

This study presented a rigorous convergence analysis of the differential transform method (DTM) for solving differential algebraic systems, addressing the challenges posed by the structural complexity and index nature of such systems. The results establish the theoretical foundation necessary for the reliable application of DTM. To enhance the convergence interval and improve practical applicability, the multi-stage differential transform method (MsDTM) was employed. Numerical examples demonstrated that MsDTM not only extends the convergence range but also provides accurate and computationally efficient solutions. Comparative results with standard DTM confirm the superiority of MsDTM in handling differential algebraic systems. These findings confirm that both DTM and MsDTM are effective, reliable, and robust tools for obtaining approximate solutions to DAEs.

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